Explicit computations for some discrete random processes.

Vladimir Dergachev

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1 Overview

In many numerical models (in particular ABIR) an important part is played by discrete-valued discrete-time random quantities that influence the evolution of the model. As the study often involves observation and comparison of behaviour influenced by processes of different kind it is important to relate the volatility of quantities of disparate nature. This note provides explicit computation for two such processes: "jump" and "walk" and compares their behaviour.

2 "Jump" and "walk" processes

We would consider two discrete-time processes that attain integer values between $[K_1, K_2]$, inclusive. We will specify how to obtain value X_{k+1} given X_k.

2.1 "Jump" process

The "jump" process is governed by single quantity v that we will call "volatility". at each point in time it can choose with probability v to pick a new value. If the decision to pick a new value has been reached it chooses a new value uniformly among those it can attain. Note that it is possible that $\tilde{X}_{k+1} = \tilde{X}_k$ even though a choice to pick a new value has been made.

2.2 "Walk" process

"Walk" process is similar to jump. At each time step it can decide to change its value with probability w (which will call the volatility of the walk process). If the choice to change value was made it will either increase it by 1 or decrease it by the same amount. However, if the new value exceed the maximum value possible - K_2 - it will be clipped to the maximum possible value. Similarly, if the value is below the smallest value attainable it is set to K_1. In both cases the actual value of the process will not change.

3 Statement of the problem

The volatility parameters for jump and walk processes do not relate to each other in a natural way. The effect that each of the processes can take on a numerical model incorporating them will depend greatly upon the nature of the model utilizing these processes. For any single relationship between volatility parameters that we might specify there would be a process that we show it to be inequivalent. However, such relationship is desirable when one wants to study the effect on the numberical model under investigation of replacing jump process with walk and vice versa under "same" volatility settings, thus exposing the sensitivity of the model to the local properties of the processes.

We propose to solve this situation by constructing a series of benchmarks that the researcher can use to measure "actual" volatility of these processes, independently from the model being studied, and thus relate different volatility parameters by the values obtained from these benchmarks.

To describe the "actual" volatility of process X_k in the time period $[T_1, T_2]$ we will use the following quantities:

$$b_{T_1,T_2}(\alpha) = \sum_{i=T_1}^{T_2-1} |X_{i+1} - X_i|^\alpha$$

We define $b_{T_1,T_2}(0)$ to be number of changes $X_i \neq X_{i+1}$. The actual benchmark of the process will be

$$B_{T_1,T_2}(\alpha) = \mathbf{E} b_{T_1,T_2}(\alpha)$$

- that is the expected value of $b_{T_1,T_2}(\alpha)$.

4 Analysis of the "jump" process

Observe that

$$\begin{aligned} B_{T_1,T_2}(\alpha) &= \mathbf{E} b_{T_1,T_2}(\alpha) = \mathbf{E} \sum_{i=T_1}^{T_2-1} |X_{i+1} - X_i|^\alpha = \sum_{i=T_1}^{T_2-1} \mathbf{E} |X_{i+1} - X_i|^\alpha = \\ &= \sum_{i=T_1}^{T_2-1} \mathbf{E} \left(\mathbf{E} \left(|X_{i+1} - X_i|^\alpha |X_i \right) \right) \end{aligned}$$

For the jump process we have

$$\mathbf{E} \left(|X_{i+1} - X_i|^\alpha |X_i \right) = \sum_{k=K_1, k \neq X_i}^{K_2} \frac{v}{K_2 - K_1 + 1} |k - X_i|^\alpha$$

The distribution of X_i can be expressed via the distribution of X_0:

$$\mathcal{F}_{X_i} = (1-v)^i \mathcal{F}_{X_0} + (1 - (1-v)^i) \mathcal{U}_{K_1,K_2}$$

where $\mathcal{U}_{K_1,K_2}$ is the uniform distribution on $[K_1, K_2]$. We see that with i tending to infinity the distribution of X_i approaches uniform. This convergence is quite fast. A rough estimate for $|\mathcal{F}_{X_i} - \mathcal{U}_{K_1,K_2}| \leq \epsilon$ is

$$i > \frac{\log \epsilon}{\log(1-v)}$$

In particular, $i > -\frac{\log \epsilon}{v}$ will suffice. Consequently, assuming T_1 and T_2 are large enough we have

$$\begin{aligned} B_{T_1,T_2}(\alpha) &\approx (T_2 - T_1) \textstyle\sum_{k,j=K_1}^{K_2} \frac{v}{(K_2-K_1+1)^2} \left|k-j\right|^\alpha = \\ &= \textstyle\frac{v(T_2-T_1)}{(K_2-K_1+1)^2} \sum_{m=1}^{K_2-K_1} 2\left(K_2 - K_1 - m + 1\right) m^\alpha \end{aligned}$$

and

$$B_{T_1,T_2}(0) \approx (T_2 - T_1)v\left(1 - \frac{1}{K_2 - K_1 + 1}\right) = \frac{v\left(K_2 - K_1\right)\left(T_2 - T_1\right)}{K_2 - K_1 + 1}$$

5 Analysis of the "walk" process

Again we use

$$\begin{aligned} B_{T_1,T_2}(\alpha) &= \mathbf{E} b_{T_1,T_2}(\alpha) = \mathbf{E}\textstyle\sum_{i=T_1}^{T_2-1} \left|X_{i+1} - X_i\right|^\alpha = \sum_{i=T_1}^{T_2-1} \mathbf{E}\left|X_{i+1} - X_i\right|^\alpha = \\ &= \textstyle\sum_{i=T_1}^{T_2-1} \mathbf{E}\left(\mathbf{E}\left(\left|X_{i+1} - X_i\right|^\alpha \left|X_i\right)\right)\right. \end{aligned}$$

For the walk process we have

$$\mathbf{E}\left(\left|X_{i+1} - X_i\right|^\alpha \left|X_i\right)\right. = \sum_{k=K_1, k\neq X_i}^{K_2} \frac{w}{K_2 - K_1 + 1} \cdot \frac{2 - \delta_{X_i,K_1} - \delta_{X_i,K_2}}{2}$$

Where $\delta_{i,j}$ is the Kronecker symbol, it is equal to 1 when $i = j$ and is 0 when $i \neq j$. Note how the result does not depend on α and is invariant with respect to transformation $[K_1, K_2] \to [K_1 + a, K_2 + a]$. Most processes that can attain finite number of values stabilize as time increases. This is also the case for the walk process. The transition matrix for it is

$$T = \begin{pmatrix} 1-\frac{w}{2} & \frac{w}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{w}{2} & 1-w & \frac{w}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{w}{2} & 1-w & \frac{w}{2} & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1-w & \frac{w}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{w}{2} & 1-w & \frac{w}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{w}{2} & 1-\frac{w}{2} \end{pmatrix}$$

The unique eigenvector with eigenvalue 1 is $\left(\frac{1}{K_2-K1+1}, \ldots, \frac{1}{K_2-K_1+1}\right)$. To estimate the rate of convergence of X_i to uniform distribution we need to bound all other eigenvalies. To do this we represent the transition matrix as:

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} + w \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Define $\tilde{A}$ as

$$\tilde{A}=\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$\tilde{A}$ is degenerate as it posseses an eigenvalue 0 with multiplicity 1. In the basis $\{(1,1,\dots,1),(-1,1,0,\dots,0),e_3\dots,e_{K_2-K_1+1}\}$, where e_i is the standard basis vector, it will have the first column and row all zeros and the only non-degenerate minor of size K_2-K_1 is A:

$$A=\begin{pmatrix} \frac{5}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Note that norm of the second basis vector is $\sqrt{2}$.

Then A^{-1} is

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{5}{3} & -\frac{5}{3} & -\frac{5}{3} & \cdots & -\frac{5}{3} & -\frac{5}{3} \\ \frac{1}{3} & -\frac{5}{3} & -\frac{11}{3} & -\frac{11}{3} & \cdots & -\frac{11}{3} & -\frac{11}{3} \\ \vdots & & & \ddots & & & \vdots \\ \frac{1}{3} & -\frac{5}{3} & -\frac{11}{3} & \cdots & \frac{19-6(K_2-K_1)}{3} & \frac{19-6(K_2-K_1)}{3} & \frac{19-6(K_2-K_1)}{3} \\ \frac{1}{3} & -\frac{5}{3} & -\frac{11}{3} & \cdots & \frac{19-6(K_2-K_1)}{3} & \frac{13-6(K_2-K_1)}{3} & \frac{13-6(K_2-K_1)}{3} \\ \frac{1}{3} & -\frac{5}{3} & -\frac{11}{3} & \cdots & \frac{19-6(K_2-K_1)}{3} & \frac{13-6(K_2-K_1)}{3} & \frac{7-6(K_2-K_1)}{3} \end{pmatrix}$$

We are now in position to estimate the norms of operators A and A^{-1}. Taking into the account that the second vector norm is $\sqrt{2}$ we have

$$|A| \leq 2$$

and

$$\left|A^{-1}\right| \leq K_2 - K_1 + 1$$

Consequently all eigenvalues, except for 1, of the transition matrix T are within $\left[1 - 2w, 1 - \frac{w}{K_2-K_1+1}\right]$.

Thus, with i tending to infinity the distribution of X_i approaches uniform. A rough estimate for $|\mathcal{F}_{X_i} - \mathcal{U}_{K_1,K_2}| \leq \epsilon$ is

$$i > \frac{\log \epsilon}{\log\left(1 - \frac{w}{K_2-K_1+1}\right)}$$

In particular, $i > -\frac{(K_2-K_1+1)\log\epsilon}{w}$ will suffice. Consequently, assuming T_1 and T_2 are large enough we have

$$B_{T_1,T_2}(\alpha) \approx (T_2-T_1) \sum_{j=K_1}^{K_2} \frac{w}{(K_2 - K_1 + 1)^2} \cdot (K_2 - K_1) = \frac{w\,(K_2 - K_1)\,(T_2 - T_1)}{K_2 - K_1 + 1}$$

6 Comparison of "walk" and "jump" benchmarks

From expressions obtained in the two previous sections we see that if one is only interested in $B_{T_1,T_2}(0)$ benchmark (i.e. the actual numerical value of the process is not important) the volatility parameters v and w are properly scaled as is.

If, however, one uses $B_{T_1,T_2}(\alpha)$ for non-zero α these parameters are best related by scaling by a constant value that depends on K_1, K_2 and α. Note that, since v and w express probability and are limited to be within $[0, 1]$, not every v or w will have a related counterpart for $\alpha \neq 0$.

$$\frac{w}{v} = \frac{\sum_{m=1}^{K_2-K_1} 2\,(K_2 - K_1 - m + 1)\, m^{\alpha}}{(K_2 - K_1)\,(K_2 - K_1 + 1)}$$

We now tabulate this relationship for some particular values of K_1, K_2 and α.

$$K_1 = -1,\ K_2 = 1$$
$$\frac{w}{v} = \frac{2^{\alpha}+2}{3}$$

α	$\frac{w}{v}$
0	1.0
0.5	1.138
1	1.333
2	2.0
3	3.333

$$K_1 = -2,\ K_2 = 1 \text{ and } K_1 = -1,\ K_2 = 2$$
$$\frac{w}{v} = \frac{3^{\alpha}+2\cdot 2^{\alpha}+3}{6}$$

α	$\frac{w}{v}$
0	1.0
0.5	1.260
1	1.667
2	3.333
3	7.667

$$K_1 = -2,\ K_2 = 2$$
$$\frac{w}{v} = \frac{4^{\alpha}+2\cdot 3^{\alpha}+3\cdot 2^{\alpha}+4}{10}$$

α	$\frac{w}{v}$
0	1.0
0.5	1.371
1	2.0
2	5.0
3	14.6

We must note that the benchmarks computed above are valid only when the distribution of X_i can be considered sufficiently close to the uniform distribution. The estimates for reaching this condition were computed in the previous sections. For example, for a jump process with $v = 0.05$ (i.e. 5%) we see that the distribution will approach uniform within 0.05 tolerance after 60 time steps. For walk process this estimate should be multiplied by the length of the range of the process: $K_2 - K_1 + 1$.